Game Theory

Lecture 6: Nash equilibrium – continued

Frieder Neunhoeffer

Lisbon School of Economics & Management
Universidade de Lisboa

UNIVERSIDADE DE LISBOA

Introduction

In this lecture, we will deal with a few topics about the concept of equilibrium solution (Nash equilibrium):

1. Theorems of existence of Nash equilibrium (in mixed and in pure strategies)
2. Types of Nash equilibrium.
3. Nash equilibria and dominant strategy equilibria.

Kakutani's Fixed Point Theorem

The theorem states that under *certain conditions*, a set-valued function or correspondence (which assigns a set of possible values instead of a single value) will have at least one **fixed point**. A fixed point is a point x^* such that:

$$x^* \in F(x^*)$$

This means that applying F to x^* does not move it anywhere outside itself.

- The strategy space of players in a finite game is **compact** and **convex** (probabilities sum to 1 and are non-negative).
- The **best-response correspondence** is upper semicontinuous and has a convex image (mixed strategies allow weighted combinations).

Therefore, by Kakutani's theorem, there exists a **fixed point**, which corresponds to a **NE**.

Condition 1

X is a non-empty, compact, and convex subset of $\mathbb{R}^n$

- <u>Non-empty</u>: The set X must contain at least one element.
- <u>Compact</u>: The set is *closed* (contains its boundary) and *bounded* (does not stretch infinitely).
- <u>Convex</u>: If you pick two points in X, the line segment between them is also in X.

Example: The unit square $X = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ in $\mathbb{R}^2$ is compact and convex.

Non-Example: $X = \{(x, y) \mid 0 \le x \le 1, 0 \le y < 0.5, 0.5 < y < 1\}$ in $\mathbb{R}^2$ is not compact because it does not include all boundaries, and it is not convex because it does not include $y = 0.5$.

Condition 2a

$F(x)$ is non-empty for all $x \in X$.

- For each point x in X, the function F must give at least one valid output.
- This ensures that F is *well-defined* and does not "break" for any x.

Example: Suppose $F(x)$ is a best-response function in a game. Every player always has at least one strategy to choose, so the best-response set is never empty.

Non-Example: If $F(x)$ sometimes returns an empty set, then we cannot guarantee the existence of a fixed point.

Condition 2b

$\boldsymbol{F(x)}$ **is convex for all** $\boldsymbol{x \in X}$**.**

- If y_1 and y_2 are in $F(x)$, then any convex combination:

$$\lambda y_1 + (1 - \lambda) y_1 \text{ for } 0 \leq \lambda \leq 1$$

  must also be in $F(x)$.

Example: In a game, if a player has two best-response strategies, they should also be able to mix between them (play a probability-weighted strategy).

Non-Example: If a certain mixed strategy is not included in the set of possible best-response strategies.

Condition 3

$F(x)$ is upper semi-continuous.

- If you slightly change x, the values in $F(x)$ do not suddenly "jump" in a crazy way.
- Formally, if $x_n \rightarrow x$, then the corresponding sequence $y_n \in F(x_n)$ must have a limit y that is in $F(x)$.

Example: If we are choosing optimal strategies in a game, and one player makes a small change in their strategy, the best-response strategies of the other players should not suddenly jump unpredictably.

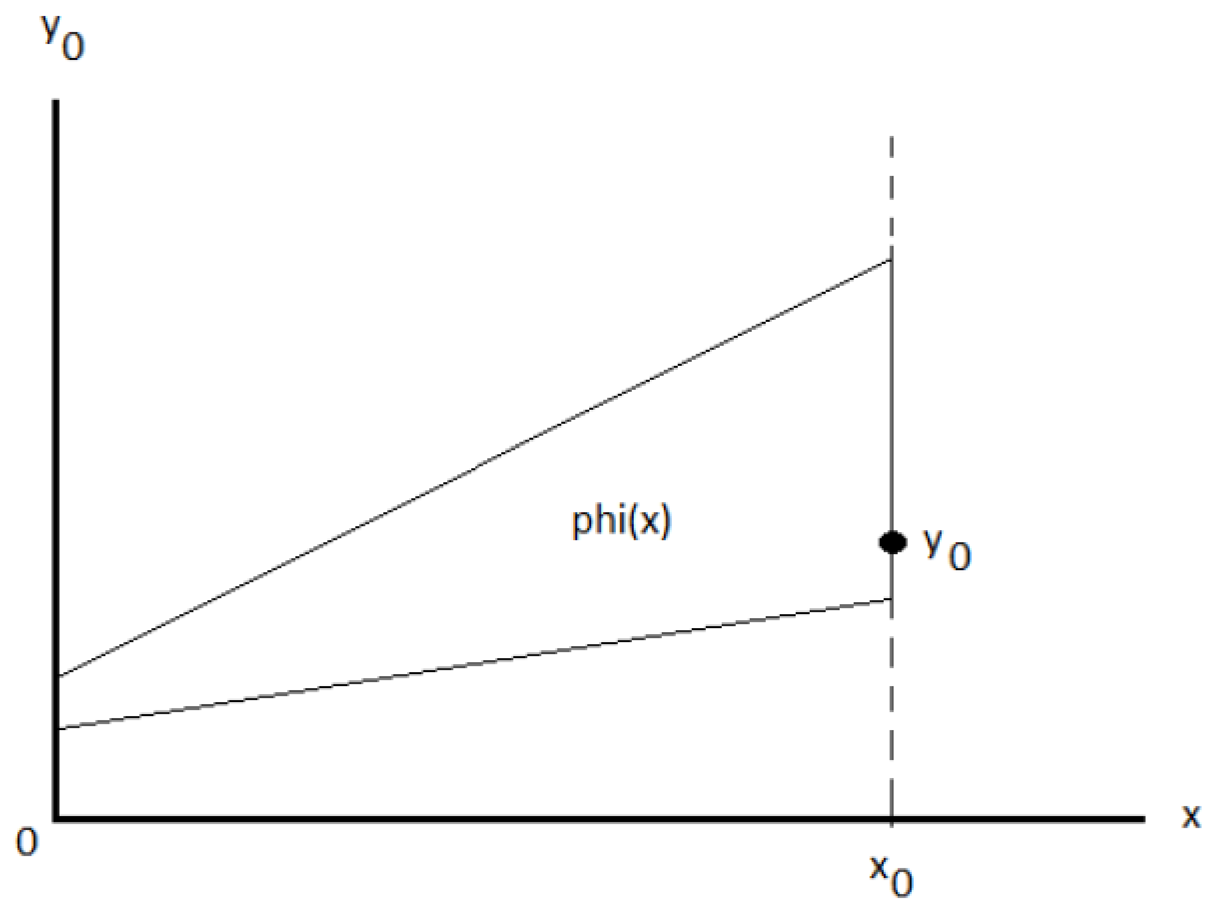
Non-Example: A function that is not upper semi-continuous could exhibit *discontinuous jumps* in best responses, making it impossible to ensure a stable equilibrium.

Upper semicontinuous correspondence

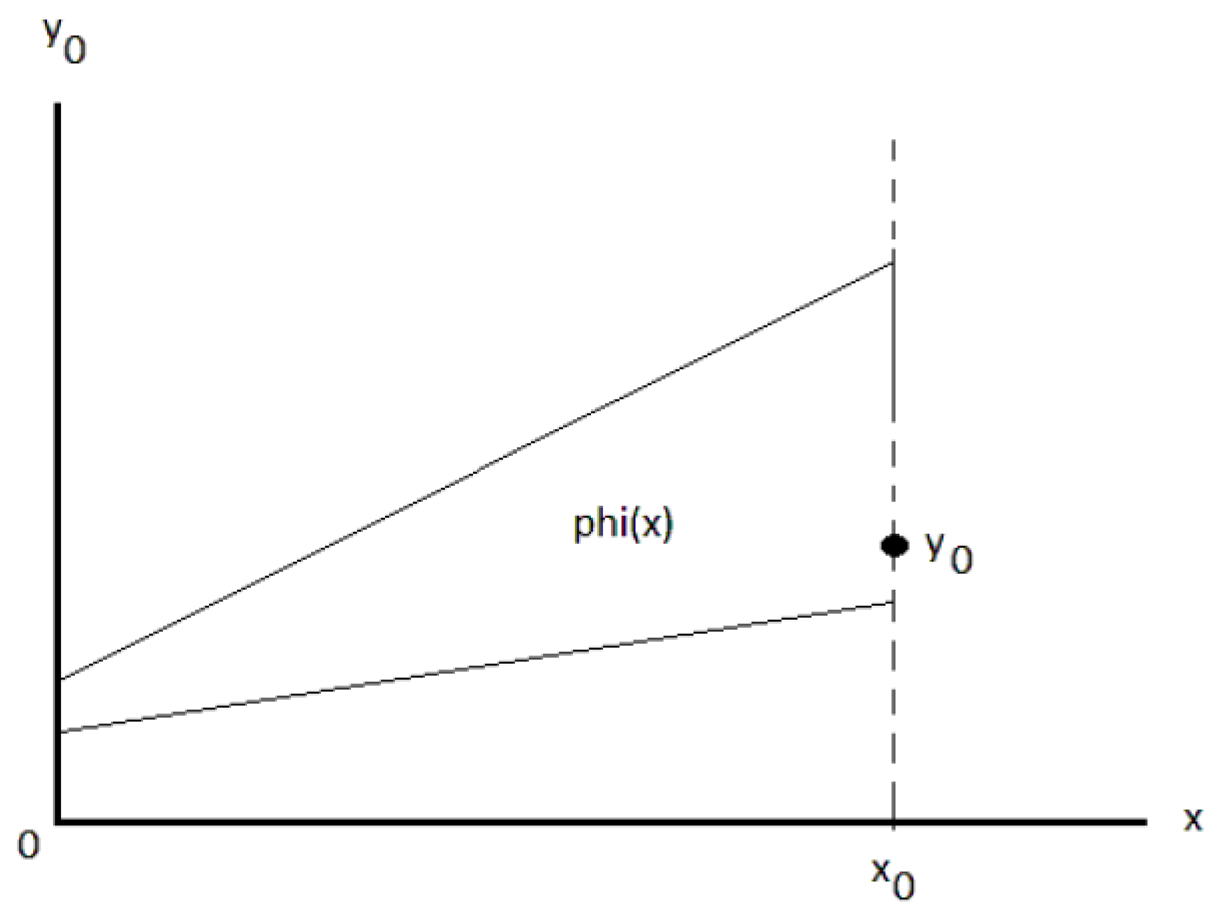
A correspondence $\phi : X \to Y$ is *upper semicontinuous* in $x_0 \in X$, if for any sequence $\{x_i\}_{i=1,2,\ldots}$ which converges to x_0 and any sequence $\{y_i\}_{i=1,2,\ldots}$ that satisfies $y_i \in \phi(x_i)$ and converges to y_0, it is always true that $y_0 \in \phi(x_0)$.

<u>Alternative definition</u>: A correspondence is *upper semicontinuous everywhere* iff its graph is a closed set, i.e., iff it contains its boundary points.

Upper semicontinuous correspondence: $y_0 \in \phi(x_0)$



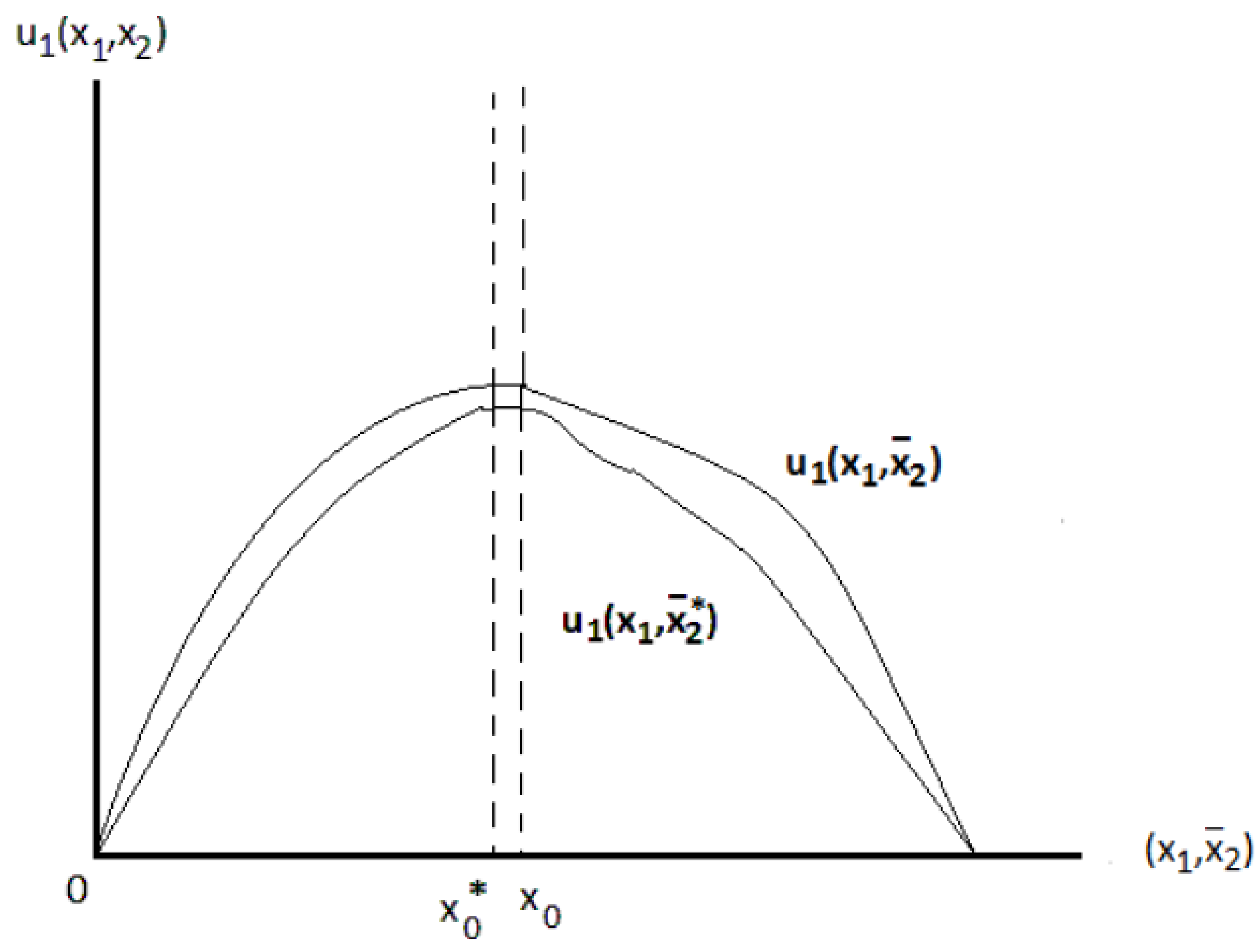
Non upper semicontinuous correspondence $\phi(x)$



Continuity of payoff function and best reply correspondence

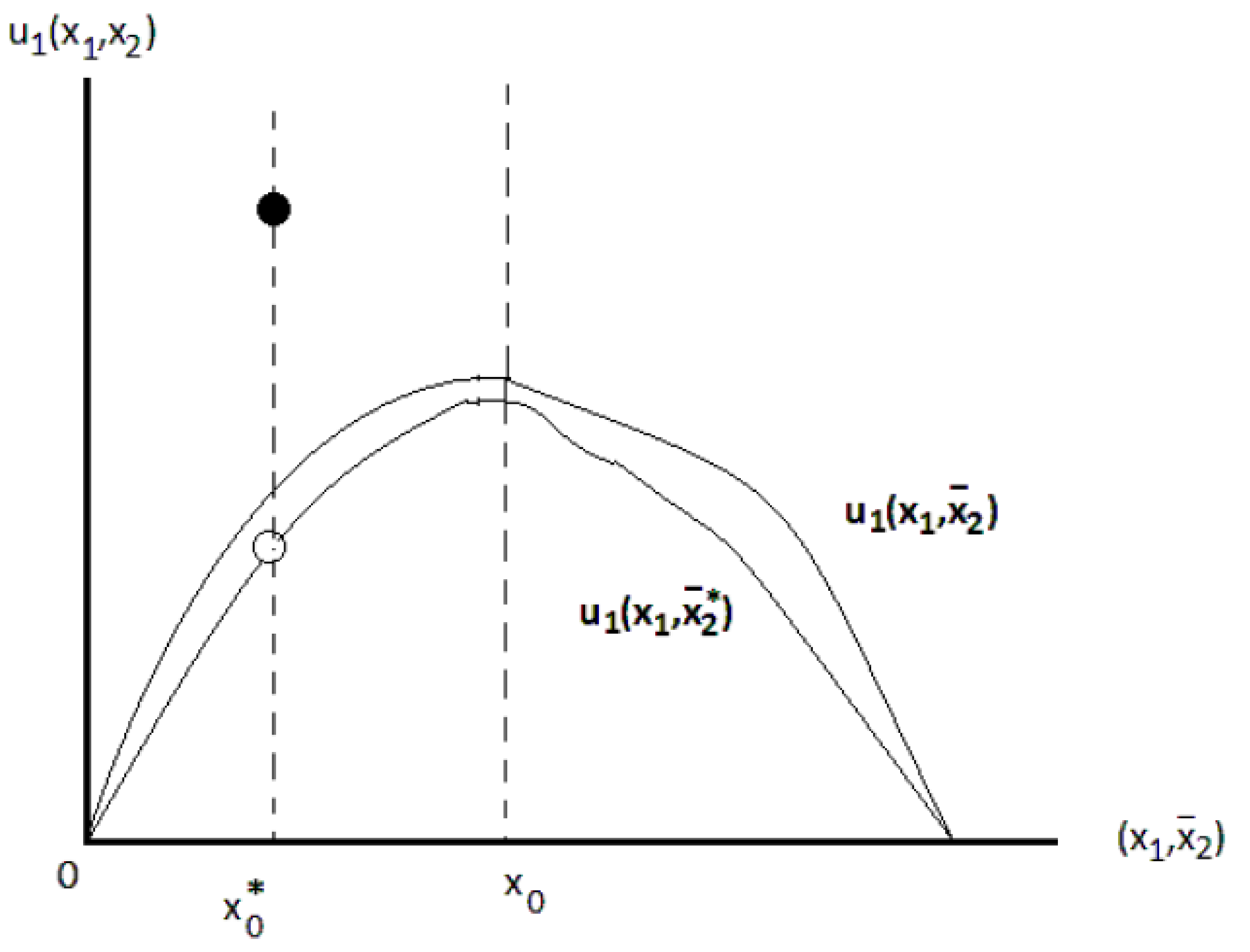
The correspondences of mixed best replies are upper semicontinuous, because the expected payoff function is a continuous function of the mixed strategies.

The figure plots the payoff function $u_1(x_1, x_2)$ of player 1 in a game with two players, where x_i $(i = 1, 2)$ are the strategies of the players. If these are mixed strategies, then $u_1(x_1, x_2)$ will be continuous as in the figure. Since the plot is two-dimensional, the figure takes the payoff as a function of x_1, x_2 being treated as a parameter that can take two arbitrarily close values $\bar{x}_2$ and $\bar{x}_2^*$. The Figure shows that, if the payoff function is continuous, the best replies of player 1 to $\bar{x}_2$ and $\bar{x}_2^*$, x_0 and x_0^*, will be arbitrarily close.



Continuity of payoff function and best reply correspondence

By contrast, this payoff function exhibits a discontinuity, so that the best reply to $\bar{x}_2^*$, x_0^*, is bounded away from the initial best reply x_0. Thus, there are points in a small neighborhood of x_0, that are boundary points of the graph of the correspondence $x_1 \in \phi(x_2)$ and do not belong to it. Consequently, the correspondence is not upper semicontinuous.



Furthermore, the image set under this kind of correspondence is always convex, because, whenever two pure strategies are best replies to a given profile of mixed strategies, every convex combination of these strategies is also a best reply.

Then, we need a theorem of existence of fixed points of correspondences.

Theorems of existence of a Nash equilibrium

Why is Nash Equilibrium More General?

Dominant Strategy Equilibrium (DSE) requires that a player's best strategy is *independent* of what the other players do. But many games do not have a dominant strategy for all players.

Rationalizable Strategies (RS) eliminate dominated strategies iteratively, but this process does not always lead to a unique or even a clear solution.

Nash Equilibrium (NE), however, always exists in games with mixed strategies (proven by Nash’s existence theorem).

Mixed strategy equilibrium

Nash Theorem: In any game with a finite number of players and a finite number of pure strategies available to each player, the set of Nash equilibria is nonempty, i.e., $\Theta^{NE} = \emptyset$.

Proof: The space of mixed strategies of the game, Θ, is a non-empty, compact and convex set, because it is the Cartesian product of unit simplices. As seen before, the best reply correspondence in mixed strategies $\tilde{\beta}$: Θ → Θ, is upper semicontinuous. Also, the image set $\tilde{\beta}(x)$ ⊂ Θ of each profile of mixed strategies x ∈ Θ is a non-empty, closed and convex set. Consequently, Θ and $\tilde{\beta}$ meet the assumptions of Kakutani's fixed point theorem so that there at least a fixed point, i.e., a profile of strategies x ∈ Θ such that $x \in \tilde{\beta}(x)$.

Pure strategy equilibrium

Although the Nash Theorem ensures the existence of a NE in mixed strategies in **all** non-cooperative games, its utility is rather *limited* in economics applications because most economists dislike mixed strategy equilibria.

Economists tend to say that economic agents select one among competing courses of action (i.e., *pure strategies*), rather than drawing randomly a strategy from a probability distribution over the basic alternatives of action.

Friedman Theorem

The following theorem by James Friedman defines *sufficient* conditions of existence of a Nash equilibrium in pure strategies, where these strategies are also *continuous*.

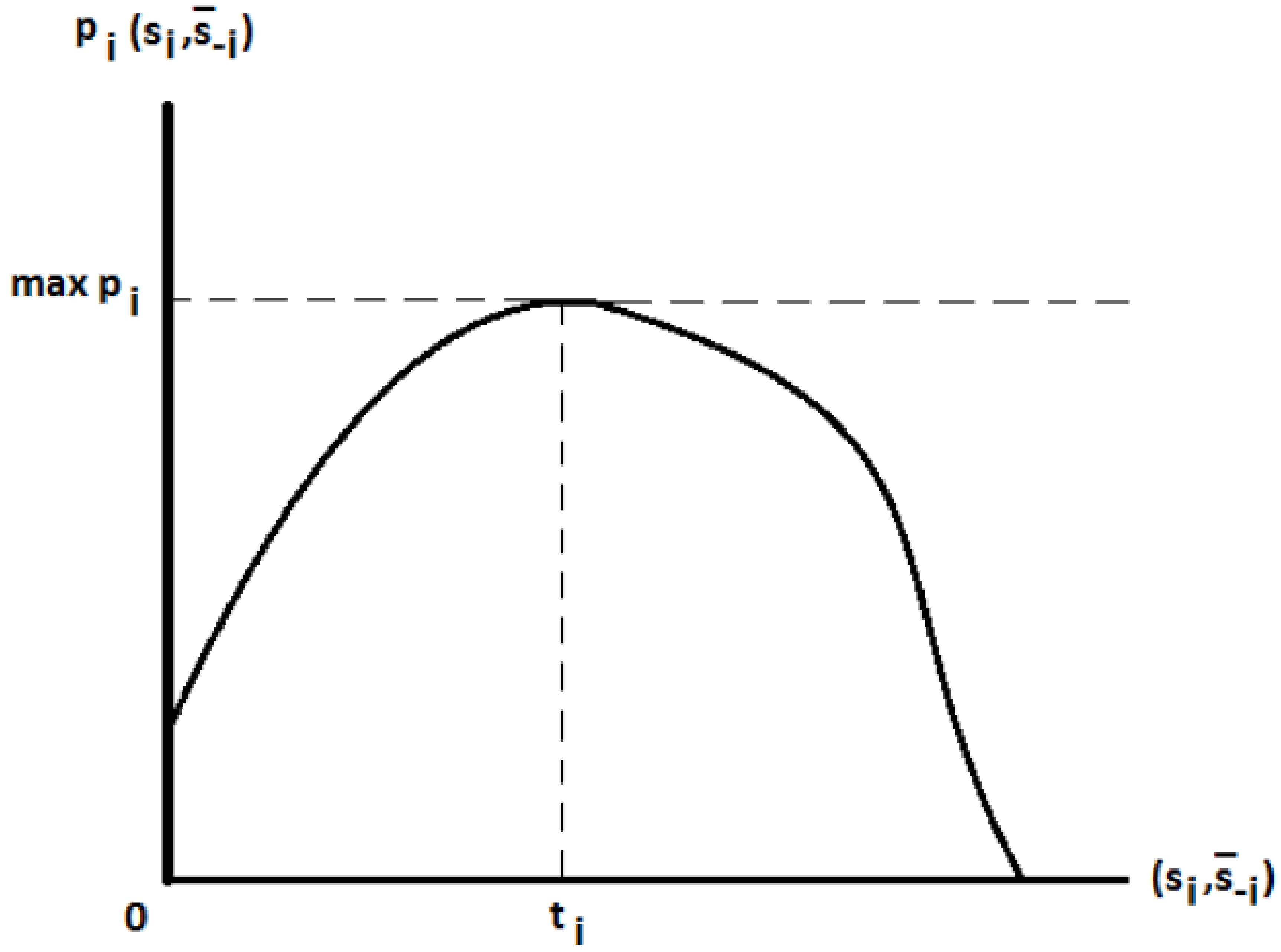
1. The number of players, n, is finite.
2. The set of pure strategies of player i, S_i, is a compact and convex subset of R^m.
3. The payoff function of player i, $\pi_i(s)$, is a scalar function that is defined for all $s = (s_1, s_2, \ldots, s_n) \in \times_i S_i = S$. This function is continuous and bounded.
4. The payoff function $\pi_i(s)$ is quasi-concave in relation to $s_i, i = 1,2,\ldots,n$.

Quasiconcavity: A function of a real variable, $f(x)$, is quasiconcave if it is:

- either monotonic
- or it is first increasing and then decreasing (the reverse being excluded)

Proof

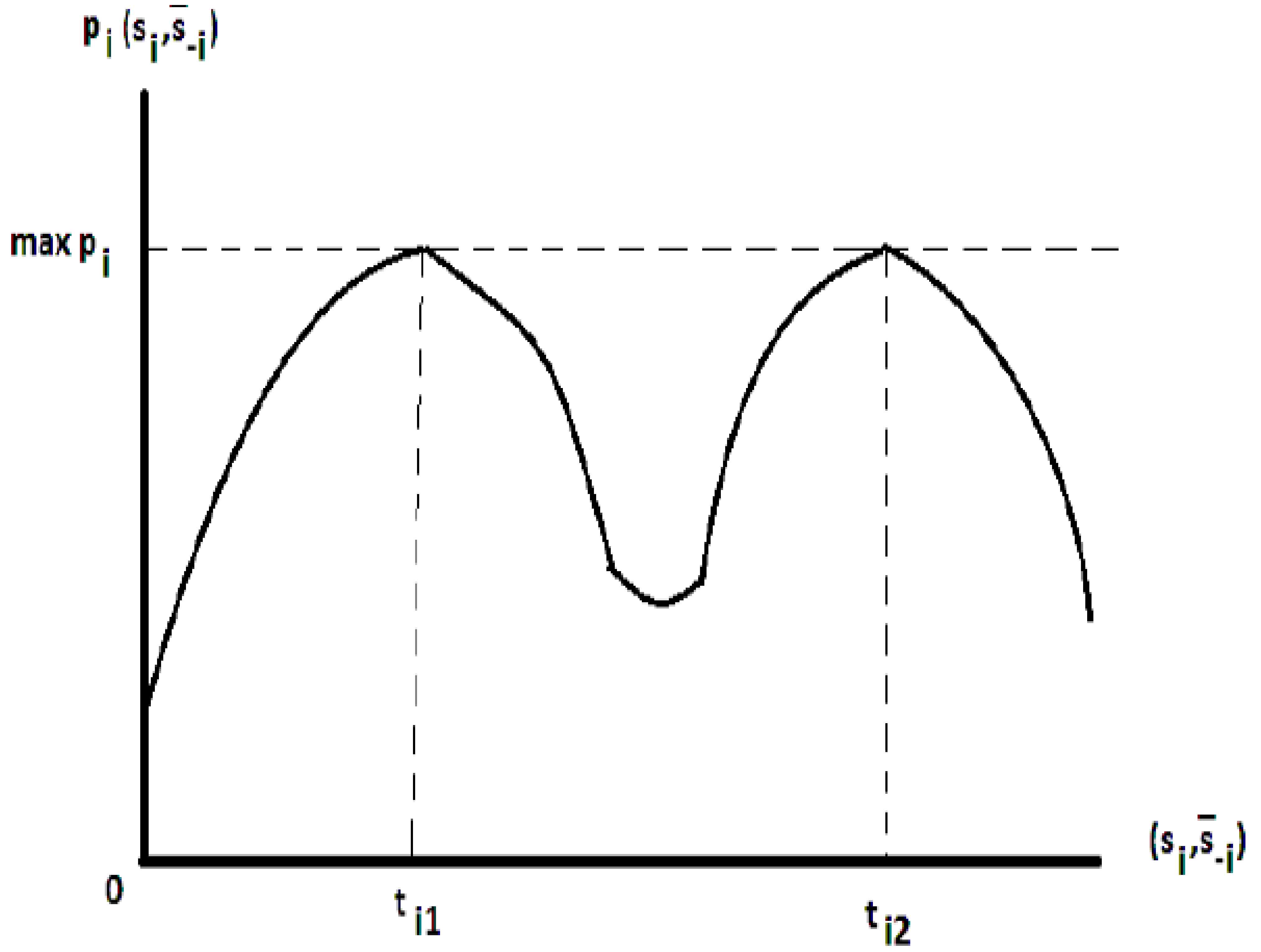
The Theorem assumptions are identical to the conditions of the fixed point (Kakutani) Theorem. Conditions 1 and 2 ensure that the space of pure $S = \times_i S_i$ is a compact and convex set. Condition 3 ensures that the best reply correspondence in pure strategies $\beta(s)$ with $s \in S$ is upper semicontinuous. Furthermore, Condition 4 determines that the image set of $\beta(s)$ is a convex set. In the figure, the payoff function $\pi_i(s)$ of player i is plotted in relation to her strategy s_i, given the profile of strategies selected by the other players s_{-i}. Since the payoff function is quasiconcave, the image set of s under the best reply correspondence of player i, $\beta_i(s)$, is a singleton $\{t_i\}$.



Proof ctd.

By contrast, if the payoff function $\pi_i(s)$ is not quasiconcave, e.g., by exhibiting two local maxima, the image set of the best reply correspondence is made by two disjoint point points $\{t_1, t_2\}$. Consequently, it is not convex and the assumptions of the Kakutani fixed point Theorem are not met.

It should be remarked that Conditions 1-4 are **sufficient**, but they are **not necessary**. A continuous game may not meet these assumptions and nevertheless have a Nash equilibrium in pure strategies.



Types of Nash equilibrium

Strict or Regular Nash equilibrium: a Nash equilibrium $x \in \Theta$ is said to be *strict* or *regular* if the strategy of each player i is a **unique** best reply to x, i.e., if

$$x = \tilde{\beta}(x)$$

While a Nash equilibrium requires that no unilateral strategy deviation by a player brings about a net gain in payoff (but there can be deviations entailing a ***zero payoff variation***), in a *strict* equilibrium, any deviation strictly decreases the player's payoff.

Hence, an arbitrarily small change in payoffs does not affect a *strict* Nash equilibrium, while it may do so in a *non-strict* equilibrium point.

Consequently, *strict* equilibria are usually regarded as more robust than *non-strict* ones.

An example

An example of a *strict* Nash equilibrium is the Nash equilibrium (Betray, Betray) with payoffs $(3,3)$ in Prisoner's Dilemma game.

$$\begin{array}{ccc} & Cooperate & Betray \\ Cooperate & 4,4 & 0,5 \\ Betray & 5,0 & 3,3 \end{array}$$

$$\begin{array}{ccc} & Cooperate & Betray \\ Cooperate & 4,4 & 0,5 \\ Betray & 5,0 & \underline{3},\underline{3} \end{array}$$

An example

However, if we change the payoff matrix to

$$\begin{array}{ccc} & Cooperate & Betray \\ Cooperate & 4,4 & 3,5 \\ Betray & 5,0 & 3,3 \end{array}$$

$$\begin{array}{ccc} & Cooperate & Betray \\ Cooperate & 4,4 & \underline{3},\underline{5} \\ Betray & 5,0 & \underline{3},\underline{3} \end{array}$$

a second NE (Cooperate, Betray) arises with payoffs $(3,5)$. Then both NE are **non-strict**, since, if the column player selects "Betray", the row player has two best replies "Cooperate" and "Betray" with payoff 3.

A NE in mixed strategies is necessarily *non-strict*, because, if a mixed strategy is a best reply for player i to a profile of strategies chosen by his opponents, then at least two pure strategies of this player are best replies either. This is the reason why most economists disregard a mixed strategy NE as a sound solution of a game.

Non-dominated Nash equilibrium

A NE strategy for player $i,\ s_i^*$, cannot be strictly dominated, but it can be weakly dominated. It is possible that there may exist another best reply, s_i', to the profile of strategies chosen in equilibrium by the opponents, s_{-i}^*, that has two properties:

- s_i' is never worse than the initial strategy s_i^* (for any s_{-i})
- There is at least a profile s_{-i} of strategies selected by the opponents such that s_i' is strictly better than s_i^*.

An example of a NE dominated strategy is "Cooperate" for the Row player in the NE (Cooperate, Betray) in the modified Prisoner’s Dilemma

	$Cooperate$	$Betray$
$Cooperate$	$4,4$	$\underline{3},\underline{5}$
$Betray$	$5,0$	$\underline{3},\underline{3}$

A NE x is said to be non-dominated if none of its components x_i is dominated. Clearly, in the modified Prisoner’s Dilemma, (Betray, Betray) is a "better“ NE than (Cooperate, Betray), because (Betray, Betray) is non-dominated.

Nash equilibrium and iterated equilibrium in non strictly dominated strategies

The relationship between these two types of equilibrium is expressed by two propositions:

Proposition 1: In a n-person game in the normal form, if the iterated elimination of strictly dominated strategies eliminates all pure strategies but the profile $s^* = (s_1^*, s_1^*, \dots, s_n^*)$ then these strategies form the *unique* pure strategy Nash equilibrium of the game.

Proposition 2: In a n-person game in normal form, if the profile strategies $s^* = (s_1^*, s_1^*, \dots, s_n^*)$ make a Nash equilibrium, then they survive the elimination of *strictly* dominated strategies.

References

- GIBBONS, Robert (1992), A Primer in Game Theory, New York, Harvester Wheatsheaf, chapter 1, section 1.3.B, pages 33-48.
- MAS-COLLEL, A. et AL. (1995), Microeconomic Theory, chapter 8, section 8.D, pages: 252-253.
- WEIBULL, Jörgen W. (1997), Evolutionary Game Theory, Cambridge Mass./London, England, chapter 1: 15-17.